MOMENTS FORMULATION OF SOME STATISTICAL PROBLEMS IN ELASTICITY

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Considered herein are some problems in elasticity with random external loads. Boundary value problem formulations are obtained involving moments of arbitrary order for the random tensor fields of stresses and displacements. For these problems, proof is given of the uniqueness theorem and of a minimum principle similar to the principle of minimum potential energy of classical elasticity.

By way of an example, the solution is obtained to the boundary value problem for second order moments of the stress tensor in the half-plane $x \geqslant 0$ in the presence of normal and shearing loads on the boundary $x = 0$, the loads being statistically homogeneous, random functions of y.

1. We consider two problems in elasticity: the displacement problem when the body forces f_i and surface loads q_i are given

$$\partial\tau_{ij} / \partial x_j = -f_i, \quad \tau_{ij}n_j = q_i \quad (x_s \in s), \qquad \tau_{ij} = c_{ijkl}\,\partial w_k / \partial x_l \tag{1.1}$$

and the stress problem when, in addition to the forces f_i and q_i, the incompatibility tensor η_{ik} is given

$$\partial\tau_{ij} / \partial x_j = -f_i, \quad \tau_{ij}n_j = q_i \quad (x_s \in s)$$

$$\varepsilon_{ijk}\varepsilon_{lmn}\frac{\partial^2 e_{km}}{\partial x_j \partial x_n} = \eta_{il}, \qquad e_{km} = s_{kmij}\tau_{ij} \tag{1.2}$$

Here τ_{ij} is the stress tensor; e_{ij} is the strain tensor; w_i is the displacement vector; c_{ijkl}, s_{ijkl} are the tensors defining the elastic properties of the medium; n_j is the normal to the body surface s; ε_{ijk} is the unit, antisymmetric, pseudotensor.

Let f_i, q_i and η_{il} be random functions of the coordinates x_s, given by their mean values and moments of various orders [1]. In view of the ordinary (with respect to w_i and τ_{ij}) and statistical (the absence of products of the random quantities) linearity of all Eqs. in (1.1) and (1.2), we may obtain separate boundary value problems for the mean values and moments of any order [2].

For the mean values, the problem statements are the same as (1.1) and (1.2), with the random functions replaced by their mean values. Such equations are also obtained for the deviations from the mean values.

For the moments of order n ($n = 1, 2, 3, \ldots$)

$$v_{i_1 \ldots i_n} = \langle v_{i_1}(x_s^1) \cdots v_{i_n}(x_s^n)\rangle, \qquad p_{i_1 j_1 \ldots i_n j_n} = \langle p_{i_1 j_1}(x_s^1) \cdots p_{i_n j_n}(x_s^n)\rangle \tag{1.3}$$

$$\gamma_{i_1 j_1 \ldots i_n j_n} = \langle \gamma_{i_1 j_1}(x_s^1) \cdots \gamma_{i_n j_n}(x_s^n)\rangle, \quad v_i = w_i - \langle w_i \rangle,$$

$$p_{ij} = \tau_{ij} - \langle \tau_{ij}\rangle, \qquad \gamma_{ij} = e_{ij} - \langle e_{ij}\rangle$$

writing each of the Eqs. in (1.1) and (1.2) for the deviations at the points $M_k(x_s^k)$, ($k = 1, \ldots, n$), multiplying and taking the means, we obtain, respectively, the boundary value problems (1.4), (1.5) and (1.4), (1.6)

$$\frac{\partial^n p_{i_1 j_1 \ldots i_n j_n}}{\partial x_{j_1}^{1} \cdots \partial x_{j_n}^{n}} = (-1)^n f_{i_1 \ldots i_n} \quad (1.4)$$

$$p_{i_1 j_1 \ldots i_n j_n} n_{j_1}(x_s^1) \cdots n_{j_n}(x_s^n) = q_{i_1 \ldots i_n} \qquad (x_s^1, \ldots, x_s^n \in s)$$

$$p_{i_1 j_1 \ldots i_n j_n} = c_{i_1 j_1 k_1 l_1} \cdots c_{i_n j_n k_n l_n} \frac{\partial^n v_{k_1 \ldots k_n}}{\partial x_{l_1} \cdots \partial x_{l_n}} \quad (1.5)$$

$$\gamma_{k_1 m_1 \ldots k_n m_n} = s_{k_1 m_1 i_1 j_1} \cdots s_{k_n m_n i_n j_n} p_{i_1 j_1 \ldots i_n j_n}$$

$$\varepsilon_{i_1 j_1 k_1} \varepsilon_{l_1 m_1 n_1} \cdots \varepsilon_{i_n j_n k_n} \varepsilon_{l_n m_n n_n} \frac{\partial^{2n} \gamma_{k_1 m_1 \ldots k_n m_n}}{\partial x_{j_1}^{1} \partial x_{n_1}^{1} \cdots \partial x_{j_n}^{n} \partial x_{n_n}^{n}} = \eta_{i_1 l_1 \ldots i_n l_n} \quad (1.6)$$

$$f_{i_1 \ldots i_n} = \langle f_{i_1}'(x_s^1) \cdots f_{i_n}'(x_s^n) \rangle, \quad q_{i_1 \ldots i_n} = \langle q_{i_1}'(x_s^1) \cdots q_{i_n}'(x_s^n) \rangle$$

$$\eta_{i_1 l_1 \ldots i_n l_n} = \langle \eta_{i_1 l_1}'(x_s^1) \cdots \eta_{i_n l_n}'(x_s^n) \rangle$$

Here, $f_{i_1 \cdots i_n}$, $q_{i_1 \cdots i_n}$ and $\eta_{i_1 l_1 \cdots i_n l_n}$ are the moments of the forces f_i, q_i and of the incompatibility tensor η_{il}. Here and hereinafter the angular brackets denote statistical means of the corresponding quantities while the primed quantities are the deviations from the means.

In connection with the boundary value problems (1.4), (1.5) and (1.4), (1.6), we note the following:

1) If, instead of (1.1), we consider the problem of given random displacements $\psi_i(x_s)$ on the boundary s, then the corresponding boundary value problem for the moments of order n will contain the first group of relations (1.4), relations (1.5) and the boundary conditions

$$v_{i_1 \ldots i_n} = \psi_{i_1 \ldots i_n}(x_s^1, \ldots, x_s^n), \quad x_s^1, \ldots, x_s^n \in s$$

$$\psi_{i_1 \ldots i_n} = \langle \psi_{i_1}'(x_s') \cdots \psi_{i_n}'(x_s^n) \rangle \quad (1.7)$$

2) Suppose that f_i, q_i, η_{iL} and ψ_i are random functions of position and slowly varying random functions of time so that a quasi-static analysis is valid. Then the above mentioned moments are defined by relations of the form

$$v_{i_1 \cdots i_n} = \langle v_{i_1}(x_s^1, t_1) \cdots v_{i_n}(x_s^n, t_n) \rangle$$

where, as before, (1.4) to (1.7) hold.

3) The boundary value problem (1.4), (1.6) includes the quasi-static theory of continuous dislocations for the case in which the dislocation density tensor has a statistical distribution. The incompatibility tensor η_{il} may be expressed in terms of the dislocation density tensor α_{lj} by means of the relation [3]

$$\eta_{il} = \varepsilon_{ijk} \partial \alpha_{lj} / \partial x_k$$

Thus, if the dislocation density α_{lj} is a random tensor field with moments

$$\alpha_{l_1 j_1 \ldots l_n j_n} = \langle \alpha_{l_1 j_1}'(x_s^1, t_1) \cdots \alpha_{l_n j_n}'(x_s^n, t_n) \rangle$$

then the stress moments are defined by problem (1.4), (1.6) with

$$\eta_{i_1 l_1 \ldots i_n l_n} = \varepsilon_{i_1 j_1 k_1} \cdots \varepsilon_{i_n j_n k_n} \frac{\partial^n \alpha_{l_1 j_1 \ldots l_n j_n}}{\partial x_{k_1} \cdots \partial x_{k_n}}$$

4) Other statistical problems in mechanics of deformable solids are reducible to problem (1.1), (1.2) namely, viscoelastic problems with random loading (problem (1.1), (1.2) is obtained by Laplace transformation of the desired functions); problems concerning the deforma-

tion of nonlinear by elastic and elasto-plastic bodies under the action of random loads (by employing the method of elastic solutions [4]); deformation problems of bodies with random inhomogeneities and random irregularity on the boundary (in solving these problems by the method of perturbations [5 to 8]).

In all of the indicated cases, the problem of determining the statistical characteristics of the stress, strain and displacement fields may be viewed in terms of the boundary value problems (1.4), (1.5) and (1.4), (1.6).

5) The set of moments (1.3) define a multi-point distribution of the random fields [9 and 10], and therefore the set of solutions of boundary value problems (1.4), (1.5) and (1.4), (1.6) statistically completely defines the fields τ_{ij}, e_{ij}, w_i.

2. The moments of the stress and strain tensors are interrelated by

$$p_{i_1j_1\ldots i_nj_n} = c_{i_1j_1k_1l_1}\cdots c_{i_nj_nk_nl_n}\gamma_{k_1l_1\ldots k_nl_n}$$

$$\gamma_{k_1l_1\ldots k_nl_n} = s_{k_1l_1i_1j_1}\cdots s_{k_nl_ni_nj_n}p_{i_1j_1\ldots i_nj_n} \qquad (2.1)$$

Let us introduce the potential function V_n of nth order moments

$$p_{i_1j_1\ldots i_nj_n} = \frac{\partial V_n}{\partial \gamma_{i_1j_1\ldots i_nj_n}}$$

By virtue of (2.1), we have the relations

$$V_n = \frac{1}{2}\,p_{i_1j_1\ldots i_nj_n}\gamma_{i_1j_1\ldots i_nj_n}, \qquad V_n = \frac{1}{2}\,p_{i_1j_1\ldots i_nj_n}\frac{\partial^n v_{i_1\ldots i_n}}{\partial x_{j_1}^{1}\cdots \partial x_{j_n}^{n}}$$

$$V_n = {}^1/_2 c_{i_1j_1k_1l_1}\cdots c_{i_nj_nk_nl_n}\gamma_{i_1j_1\ldots i_nj_n}\gamma_{k_1l_1\ldots k_nl_n}$$

$$V_n = {}^1/_2 s_{i_1j_1k_1l_1}\cdots s_{i_nj_nk_nl_n}p_{i_1j_1\ldots i_nj_n}p_{k_1l_1\ldots k_nl_n}$$

$$\gamma_{i_1j_1\ldots i_nj_n} = \frac{\partial V_n}{\partial p_{i_1j_1\ldots i_nj_n}}$$

Let us assume that V_n is a positive-definite form in its arguments. The potential function will be a homogeneous quadratic form of 6^n variables $\gamma_{i_1j_1\ldots i_nj_n}$; every coefficient in the form is a product of elastic constants of the material under consideration with the sum of the exponents in the product equal to n; the coefficients also contain numerical factors. Hence, the positive definite character of V_n may be verified, but this involves an extremely laborious investigation.

In the case of an isotropic body, for example, for V_2, we have

$$V_2 = {}^1/_2\lambda^2\gamma_{iikk}\gamma_{ppss} + \lambda\mu\,(\gamma_{iikl}\gamma_{ppkl} + \gamma_{iikk}\gamma_{ijss}) + 2\mu^2\gamma_{ijkl}\gamma_{ijkl} \qquad (2.2)$$

where λ and μ are the Lamé constants. The usual conditions

$$\mu > 0, \qquad 3\lambda + 2\mu > 0$$

guarantee the positive-definiteness of form (2.2).

Thus, let the moment potential V_n be a positive definite form. Then the uniqueness theorem holds for the boundary value problems (1.4), (1.5) and (1.4), (1.6). We will prove it for (1.4), (1.5).

Consider two solutions

$$v^{(1)}_{k_1\ldots k_n}, \quad p^{(1)}_{i_1j_1\ldots i_nj_n}; \qquad v^{(2)}_{k_1\ldots k_n}, \quad p^{(2)}_{i_1j_1\ldots i_nj_n}$$

for problem (1.4), (1.5) with the same functions $f_{i_1\ldots i_n}$, $q_{i_1\ldots i_n}$. The difference of these solutions satisfies the homogeneous differential equations with homogeneous boundary conditions

$$\frac{\partial^n p_{i_1j_1\ldots i_nj_n}}{\partial x_{j_1}^{1}\cdots \partial x_{j_n}^{n}} = 0$$

$$p_{i_1j_1\ldots i_nj_n}n_{j_1}(x_s^1)\cdots n_{j_n}(x_s^n) = 0, \quad x_s^1, \ldots, x_s^n \in s \qquad (2.3)$$

with

$$V_n = \frac{1}{2} p_{i_1 j_1 \ldots i_n j_n} \frac{\partial^n v_{i_1 \ldots i_n}}{\partial x_{j_1}^{1} \cdots \partial x_{j_n}^{n}} \tag{2.4}$$

Utilizing (2.4) and applying the Ostrogradskii-Gauss formula to each group of variables $x_s^k (k = 1, \ldots, n)$, we find

$$2 \int_{(v)} \cdots \int V_n dv_1 \cdots dv_n = - \int_{(v)} \cdots \int v_{i_1 \ldots i_n} \frac{\partial^n p_{i_1 j_1 \ldots i_n j_n}}{\partial x_{j_1}^{1} \cdots \partial x_{j_n}^{n}} dv_1 \cdots dv_n +$$

$$+ \int_{(s)} \cdots \int p_{i_1 j_1 \ldots i_n j_n} n_{j_1}(x_s^1) \cdots n_{j_n}(x_s^n) v_{i_1 \ldots i_n} ds_1 \cdots ds_n \tag{2.5}$$

In view of (2.3), (2.5) may be written as

$$\int_{(v)} \cdots \int V_n dv_1 \cdots dv_n = 0$$

Whence, by virtue of the positive-definiteness of V_n, it follows that $V_n = 0$ and

$$p_{i_1 j_1 \ldots i_n j_n} = 0, \qquad \gamma_{i_1 j_1 \ldots i_n j_n} = 0, \qquad \frac{\partial^n v_{i_1 \ldots i_n}}{\partial x_{j_1}^{1} \cdots \partial x_{j_n}^{n}} = 0$$

Thus, the boundary value problem (1.4), (1.5) uniquely determines the moments of the stresses and strains while the displacements are determined up to the solutions of Eqs.

$$\frac{\partial^n v_{i_1 \ldots i_n}}{\partial x_{j_1}^{1} \cdots \partial x_{j_n}^{n}} = 0$$

For boundary conditions (1.7), the displacement moments obtained are single-valued.

From the foregoing, it is evident that there exists a clearly defined analogy between problems (1.4), (1.5) and (1.4), (1.6) and elasticity problems (1.1), (1.2) with determinate functions f_i, q_i, η_i, which is only natural. This analogy may be extended, and we can prove for problems (1.4), (1.5) and (1.4), (1.6) a whole series of theorems which are known in elasticity. As an example, we will formulate and prove for problem (1.4), (1.5) the minimum principle which is analogous to the principle of minimum potential energy.

Consider the functional

$$W_n [v_{i_1 \ldots i_n}] = \int_{(v)} \cdots \int [V_n (\gamma_{i_1 j_1 \ldots i_n j_n}) + (-1)^n f_{i_1 \ldots i_n} v_{i_1 \ldots i_n}] dv_1 \cdots dv_n -$$

$$- \int_{(s)} \cdots \int q_{i_1 \ldots i_n} v_{i_1 \ldots i_n} ds_1 \cdots ds_n \tag{2.6}$$

The following minimum principle holds: The functional (2.6), considered as a functional on the admissible (in the sense of smoothness) moment fields $v_{i_1} \ldots v_{i_n}$, attains an absolute minimum on the real fields satisfying (1.4), (1.5).

We will prove this principle. Let $v_{i_1} \ldots v_{i_n}$ be real and $v_{i_1 \ldots i_n} + \Delta v_{i_1 \ldots i_n}$ be arbitrary admissible displacement moments. Then, since V_n is a homogeneous quadratic form, we have

$$V_n (\gamma_{i_1 j_1 \ldots i_n j_n} + \Delta \gamma_{i_1 j_1 \ldots i_n j_n}) =$$

$$= V_n (\gamma_{i_1 j_1 \ldots i_n j_n}) + V_n (\Delta \gamma_{i_1 j_1 \ldots i_n j_n}) + \frac{\partial V_n}{\partial \gamma_{i_1 j_1 \ldots i_n j_n}} \Delta \gamma_{i_1 j_1 \ldots i_n j_n} \tag{2.7}$$

and, therefore

$$W_n [v_{i_1 \ldots i_n} + \Delta v_{i_1 \ldots i_n}] - W_n [v_{i_1 \ldots i_n}] = \int_{(v)} \cdots \int V_n (\Delta \gamma_{i_1 j_1 \ldots i_n j_n}) dv_1 \cdots dv_n + K$$

$$K=\int_{(v)}\cdots\int p_{i_1j_1\ldots i_nj_n}\Delta\gamma_{i_1j_1\ldots i_nj_n}dv_1\cdots dv_n+$$

$$+\int_{(v)}\cdots\int(-1)^n f_{i_1\ldots i_n}\Delta v_{i_1\ldots i_n}dv_1\cdots dv_n-\int_{(s)}\cdots\int q_{i_1\ldots i_n}\Delta v_{i_1\ldots i_n}ds_1\cdots ds_n \quad (2.8)$$

Utilizing the relation

$$p_{i_1j_1\ldots i_nj_n}\Delta\gamma_{i_1j_1\ldots i_nj_n}=p_{i_1j_1\ldots j_nj_n}\frac{\partial^n\Delta v_{i_1\ldots i_n}}{\partial x_{j_1}^{1}\cdots\partial x_{j_n}^{n}}$$

and transforming the first term in (2.8) in the same manner as before, we obtain

$$K=-\int_{(v)}\cdots\int\Big(\frac{\partial^n p_{i_1j_1\ldots i_nj_n}}{\partial x_{j_1}^{1}\cdots\partial x_{j_n}^{n}}-(-1)^n f_{i_1\ldots i_n}\Big)\Delta v_{i_1\ldots i_n}dv_1\cdots dv_n+$$

$$+\int_{(s)}\cdots\int[p_{i_1j_1\ldots i_nj_n}n_{j_1}(x_s^1)\cdots n_{j_n}(x_s^n)-q_{i_1\ldots i_n}]\Delta v_{i_1\ldots i_n}ds_1\cdots ds_n \qquad (2.9)$$

In view of (1.4), (1.5), (2.9) yields $K = 0$. Hence, if the functions $\Delta\gamma_{i_1j_1\ldots i_nj_n}$ are not identically zero, then by virtue of the positive-definiteness of V_n, (2.7) yields

$$W_n[v_{i_1\ldots i_n}+\Delta v_{i_1\ldots i_n}]-W_n[v_{i_1\ldots i_n}]=\int_{(v)}\cdots\int V_n(\Delta\gamma_{i_1j_1\ldots i_nj_n})dv_1\cdots dv_n>0$$

which proves the previously stated minimum principle.

3. As an example in solving boundary value problems of the type (1.4), (1.6), consider the deformation of a half-plane under the action of random loads applied on its boundary.

For the plane problem involving random surface loads q_1, q_2, the boundary value problem for the second order stress moments p_{ijkl} may be written in the form

$$\nabla_1^4\nabla_2^4\Phi=0 \qquad (\Phi=\langle F'(x_1,\ y_1)F'(x_2,\ y_2)\rangle)$$

$$p_{ijkl}n_j(M_1)n_l(M_2)=q_{ik}, \qquad M_1,M_2\in L \qquad (3,1)$$

Here, q_{ik} are the second order moments of the loads q_1 and q_2; L is the boundary; ∇_k^2 is the Laplacian with respect to the point coordinates $M_k(x_k, y_k)$; Φ is the second order moment of the stress function.

In (3.1), the subscripts range over the values 1, 2. The quantities p_{ijkl} are given in terms of the stress function Φ by the relations

$$p_{1111}=\frac{\partial^4\Phi}{\partial y_1^2\partial y_2^2},\qquad p_{2222}=\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2},\qquad p_{1212}=\frac{\partial^4\Phi}{\partial x_1\partial y_1\partial x_2\partial y_2}$$

$$p_{1122}=\frac{\partial^4\Phi}{\partial y_1^2\partial x_2^2},\qquad p_{2211}=\frac{\partial^4\Phi}{\partial x_1^2\partial y_2^2},\qquad p_{1112}=-\frac{\partial^4\Phi}{\partial y_1^2\partial x_2\partial y_2}$$

$$p_{1211}=-\frac{\partial^4\Phi}{\partial x_1\partial y_1\partial y_2^2},\qquad p_{2212}=-\frac{\partial^4\Phi}{\partial x_1^2\partial x_2\partial y_2},\qquad p_{1222}=-\frac{\partial^4\Phi}{\partial x_1\partial y_1\partial x_2^2}$$

Consider the deformation problem for the half-plane $x\geqslant 0$ under the action of normal and shearing loads q_1 and q_2, respectively, which are uncorrelated, statistically homogeneous, random functions of the y coordinate.

In that case,

$$q_{11}=q_{11}(\eta),\qquad q_{22}=q_{22}(\eta),\quad q_{12}=q_{21}=0,\qquad \eta=y_2-y_1$$

The differential equation for $\Phi(x_1, x_2, \eta)$ is then given by

$$\frac{\partial^8\Phi}{\partial x_1^4\partial x_2^4}+2\frac{\partial^6}{\partial\eta^2\partial x_1^2\partial x_2^2}(\Delta\Phi)+\frac{\partial^4}{\partial\eta^4}(\Delta\Delta\Phi)+2\frac{\partial^4}{\partial\eta^4}\Big(\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2}\Big)+$$

$$+2\frac{\partial^6}{\partial\eta^6}(\Delta\Phi)+\frac{\partial^8\Phi}{\partial\eta^8}=0 \qquad (3.2)$$

where Δ is the Laplacian with respect to x_1, x_2, and (for $x_1 = x_2 = 0$) the boundary conditions are

$$p_{1111} = q_{11}(\eta), \qquad p_{2121} = q_{22}(\eta), \quad p_{1121} = p_{2111} = 0 \tag{3.3}$$

Let q_{11} and q_{22} be given by the spectral representation

$$q_{ik}(\eta) = \int_{-\infty}^{\infty} s_{ik}(\lambda) e^{i\lambda\eta} d\lambda$$

and let us seek a solution to the problem (3.2), (3.3) in the form

$$\Phi(x_1, x_2, \eta) = \int_{-\infty}^{\infty} U(x_1, x_2, \lambda) e^{i\lambda\eta} d\lambda$$

$$p_{ijkl}(x_1, x_2, \eta) = \int_{-\infty}^{\infty} \tau_{ijkl}(x_1, x_2, \lambda) e^{i\lambda\eta} d\lambda \tag{3.4}$$

Then we obtain a differential equation for $U(x_1, x_2, \lambda)$ in the form

$$\frac{\partial^8 U}{\partial x_1^4 \partial x_2^4} - 2\lambda^2 \frac{\partial^4}{\partial x_1^2 \partial x_2^2}(\Delta U) + \lambda^4 \Delta\Delta U + 2\lambda^4 \frac{\partial^4 U}{\partial x_1^2 \partial x_2^2} - 2\lambda^6 \Delta U + \lambda^8 U = 0 \tag{3.5}$$

and the boundary conditions (for $x_1 = x_2 = 0$)

$$\tau_{1111} = s_{11}(\lambda), \qquad \tau_{2121} = s_{22}(\lambda), \qquad \tau_{1121} = \tau_{2111} = 0 \tag{3.6}$$

where τ_{ijkl} are given in terms of U by

$$\tau_{1111} = \lambda^4 U, \quad \tau_{2222} = \frac{\partial^4 U}{\partial x_1^2 \partial x_2^2}, \quad \tau_{1212} = \lambda^2 \frac{\partial^2 U}{\partial x_1 \partial x_2}$$

$$\tau_{1122} = -\lambda^2 \frac{\partial^2 U}{\partial x_2^2}, \quad \tau_{1112} = i\lambda^3 \frac{\partial U}{\partial x_2}, \quad \tau_{2212} = -i\lambda \frac{\partial^3 U}{\partial x_1^2 \partial x_2}$$

$$\tau_{2211}(x_1, x_2, \lambda) = \tau_{1122}(x_2, x_1, -\lambda)$$

$$\tau_{1211}(x_1, x_2, \lambda) = \tau_{1112}(x_2, x_1, -\lambda)$$

$$\tau_{1222}(x_1, x_2, \lambda) = \tau_{2212}(x_2, x_1, -\lambda)$$

The solution of problem (3.5), (3.6) takes the form

$$U = [a + b(x_1 + x_2) + c x_1 x_2] e^{-|\lambda|(x_1+x_2)}$$

$$a = \frac{s_1(\lambda)}{\lambda^4}, \quad b = \frac{|\lambda|}{\lambda^4} s_1(\lambda), \quad c = \frac{s_1(\lambda) + s_2(\lambda)}{\lambda^2}, \quad s_1 = s_{11}, \ s_2 = s_{22}$$

The stress moments (3.4) are given by the relations

$$p_{1111} = T_1^0 + (x_1 + x_2) T_1^1 + x_1 x_2 (T_1^2 + T_2^2)$$

$$p_{2222} = T_1^0 + 4T_2^0 - (x_1 + x_2)(T_1^1 + 2T_2^1) + x_1 x_2 (T_1^2 + T_2^2)$$

$$p_{1212} = T_2^0 - (x_1 + x_2) T_2^1 + x_1 x_2 (T_1^2 + T_2^2)$$

$$p_{1122} = T_1^0 + x_1 (T_1^1 + 2T_2^1) - x_2 T_1^1 - x_1 x_2 (T_1^2 + T_2^2) \tag{3.7}$$

$$p_{1112} = -x_1 R_2^1 + x_2 R_1^1 + x_1 x_2 (R_1^2 + R_2^2)$$

$$p_{2212} = -2R_2^0 + x_1 R_2^1 + x_2 (R_1^1 + 2R_2^1) - x_1 x_2 (R_1^2 + R_2^2)$$

$$p_{2211}(x_1, x_2, \eta) = p_{1122}(x_2, x_1, -\eta), \quad p_{1211}(x_1, x_2, \eta) = p_{1112}(x_2, x_1, -\eta)$$

$$p_{1222}(x_1, x_2, \eta) = p_{2212}(x_2, x_1, -\eta)$$

The expressions for T_i^k and R_i^k are of the form

$$T_i^k = 2\int_0^{\infty} \lambda^k s_i(\lambda) \cos(\lambda\eta) e^{-\lambda(x_1+x_2)} d\lambda, \quad R_i^k = 2\int_0^{\infty} \lambda^k s_i(\lambda) \sin(\lambda\eta) e^{-\lambda(x_1+x_2)} d\lambda$$

$$(i = 1, 2; \quad k = 0, 1, 2) \tag{3.8}$$

Consider two particular cases of the problem under consideration, for which solutions have been obtained by other methods.

1) The problem of stress concentration resulting from surface unevenness for the half-plane having an uneven boundary $x = \varepsilon\Delta(y)$, where ε is a small parameter and $\Delta(y)$ is a stationary random function, and subjected to a tensile stress σ in the y direction is reducible to the problem of a half-plane $x \geqslant 0$ subjected to random shearing loads of intensity $q_2 = \sigma d\Delta/dy$ on the boundary $x = 0$ [11].

Setting $s_1 = 0$ and $2s_2 = \sigma^2 s(\lambda)$, where $s(\lambda)$ is the spectral density of the random function $d\Delta/dy$, then the expression for the moment

$$B(\eta) = p_{2222}\,|_{x_1=x_2=0}$$

which was obtained in [11] by a different method, is found here, with the aid of (3.7) and (3.8), as

$$B(\eta) = 4\sigma^2 \int_0^\infty s(\lambda)\cos(\lambda\eta)\,d\lambda$$

2) In [12], the solution is obtained to the problem of a half-plane subjected to the action of a normal random load with δ-correlation ("white noise" type of load).

Setting $s_1 =$ const and $s_2 = 0$ in (3.7) and (3.8), we obtain the relations which were obtained in [12] by a different method.

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